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Extension of the Method of Direct Separation of Motions for Problems of Oscillating Action on Dynamical Systems

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Abstract

A general approach to study oscillating action on nonlinear dynamical systems is developed. It implies a transition from initial governing equations of motion to much more simple equations describing only the main slow component of motions (the vibro-transformed dynamics equations). The approach is named as the Oscillatory Strobodynamics, since motions are perceived as under a stroboscopic light. The vibro-transformed dynamics equations comprise terms that represent the averaged effect of the oscillating action. The method of direct separation of motions (MDSM) appears to be an efficient and simple tool to derive these equations. A modification of the method applicable to study problems that do not imply restrictions on the spectrum of excitation frequencies is proposed. It allows also to abandon other restrictions usually introduced when employing the classical asymptotic methods, e.g. the requirement for the involved nonlinearities to be weak.
Several relevant examples from Mechanics, Physics, Chemistry, and Biophysics are considered by means of the conventional MDSM and, in more details, by the modified MDSM, illustrating the efficiency the methods.

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Keywords: Oscillating action; nonlinear dynamical systems; slow motion; small parameter; the method of direct separation of motions.

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1. Introduction

Oscillating action on nonlinear dynamical systems gives rise to several unusual, seemingly paradoxical phenomena[1,2]. Stabilization of pendulum upper position, vibrational maintaining or braking of rotations, changing of materials rheological properties, excitation or suppression of chaotic motions, and many other effects can be mentioned here (see e.g.[1-4]). These phenomena in some cases can be employed to improve existing technological processes and machines, in others, in contrary, lead to accidents and even catastrophes. Such phenomena arising in the field of mechanics are relatively well studied, in particular by means of the general approach, named Vibrational Mechanics[1] (VM), and the corresponding analytical method, the Method of Direct Separation of Motions (MDSM). Analysis of oscillating action on physical, chemical, biological systems and production processes is, however, just getting started. Only several studies conducted by different, mostly numerical, methods are published (see, e.g.[5,6]). On the other hand, it is hard to indicate a phenomenon or a process for which effects of oscillating action are of no practical or scientific interest.

The first aim of the present work is to extend the VM approach and the MDSM for studying dynamical systems from various fields of science, e.g. physics, chemistry, biology and others. The basic idea of such extension has been discussed in the papers[7,8,9]. The section of nonlinear dynamics studying oscillating actions on dynamical systems was proposed to be named as the Oscillatory Strobodynamics (OS). The name is motivated by the fact that the corresponding general approach implies system behavior to be perceived as under a stroboscopic light, so that only the main, slow component of motions is captured.

Most of the problems considered by the MDSM can be solved also by the other methods of nonlinear dynamics, e.g. the multiple scales method[10] or the method of averaging[11,12]. However, the MDSM features several significant advantages over these methods, e.g. the simplicity in application and the transparency of the physical interpretation. A detailed comparison of the MDSM with the other methods is given in the monographs[1,2].

The conventional MDSM implies frequency of oscillating action to be high, i.e. much larger than the system characteristic (natural) frequency. The classical asymptotic methods, e.g. the multiple scales method[10] and the method of averaging[11,12], also imply restrictions on the excitation frequency spectrum: only near-resonant, low-frequency or high-frequency excitations can be captured. The second aim of the present work is to develop a modification of the MDSM for solving a broader range of problems, namely problems that do not imply restrictions on the spectrum of excitation frequencies. Such a modification is especially relevant for continuous systems having multitude of natural frequencies. This version of the MDSM allows also abandoning other restrictions usually introduced when employing the classical asymptotic methods, e.g. the requirement for the involved nonlinearities to be weak. So, problems without an explicit small parameter can be considered by means of the method. The idea of such modification of the MDSM has been discussed in the papers[9,13,14]. It is closely related to the main aim of the paper discussed above, since such modification of the method is particularly relevant for problems arising in electrical engineering, physics, chemistry, biology etc.

In the present paper, the proposed extension of the VM and the modification of the MDSM are illustrated by several relevant examples.

2. Slow and fast motions of nonlinear systems under high-frequency excitations; the main idea of the OS approach

First, we consider high-frequency external excitations. Motion $\boldsymbol{x}(t)$ of a dynamical system arising due to such excitation can be usually separated into two components: slow $\boldsymbol{X}(t)$ and fast $\boldsymbol{\psi}(t)$ (notions “high-frequency”, “fast” and “slow” can be formalized[1]). Fig. 1 illustrates this for the simple one-dimensional case. Note that exactly the same figure illustrates the well-known definition of oscillations as a process described by the coordinate $x(t)$ which from time to time intersects a certain constant or slowly varying level[15]. Thus the above statement is rather general: it merely means that the system under oscillating action exhibits oscillations.

The main idea of the proposed approach lies in the transition from initial governing equations of motions to equations describing only the slow component $\boldsymbol{X}(t)$. This component is usually of primary interest; and equations describing it can be much simpler than the initial equations for vector $\boldsymbol{x}(t)$.

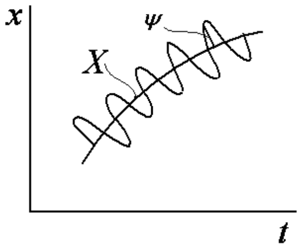

Fig. 1. Fast and slow motions of an oscillating dynamical system; definition of oscillations.

Let dynamics of a process be described by the relation:

$$\boldsymbol{Z}\left(\boldsymbol{x},\boldsymbol{a},t\right)=0\,, \tag{1}$$

where $\boldsymbol{x}$ is state vector of the considered system, $\boldsymbol{a}$ vector of parameters, t time. Operator $\boldsymbol{Z}$ can represent finite, differential, integral and other equations with the corresponding initial and boundary conditions. In the presence of high-frequency excitation this relation takes the form:

$$\boldsymbol{Z}[\boldsymbol{x}+\boldsymbol{\psi}_x(\omega t),\boldsymbol{a}+\boldsymbol{\psi}_a(\omega t),t]=\boldsymbol{F}(\omega t)\,, \tag{2}$$

where ψ_x, ψ_a and $\boldsymbol{F}$ are functions periodic in the "fast time" $\tau=\omega t$, and $\omega >> 1$. It is assumed therefore that the high-frequency excitation can be imposed directly on the state vector $\boldsymbol{x}$, or the vector of parameters $\boldsymbol{a}$; external excitation $\boldsymbol{F}$ is also possible.

Practically in all cases the change of the vector $\boldsymbol{x}$ due to high-frequency actions can be represented as:

$$\boldsymbol{x}(t)=\boldsymbol{X}(t)+\boldsymbol{\psi}(t,\omega t)\,, \tag{3}$$

where $\boldsymbol{X}$ is slow, and $\boldsymbol{\psi}$ is fast 2π-periodic in time $\tau=\omega t$ variable, with average zero:

$$\left\langle \boldsymbol{\psi} \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{\psi}(t,\tau)d\tau = 0\,; \tag{4}$$

angular brackets designate averaging in the period 2π on time τ. Variable $\boldsymbol{X}(t)$, as it was noted above, is of primary interest; and relations (3)-(4) represent the assumption that periodic oscillations arise in the system due to high-frequency excitations.

By means of one mathematical method or another, and with averaging procedure being employed, under certain assumptions regarding the operator $\boldsymbol{Z}$, it is possible to obtain equation that involves only the slow component $\boldsymbol{X}$:

$$\boldsymbol{Z}^*(\boldsymbol{X},\boldsymbol{a}^*,t)=0\,. \tag{5}$$

Note that operator $\boldsymbol{Z}^*$ is much simpler than $\boldsymbol{Z}$, e.g. it can be of a lower dimension. The same applies for the vectors of parameters $\boldsymbol{a}$ and $\boldsymbol{a}^*$.

So, slow motions $\boldsymbol{X}$ of the system obey dynamic laws that differ from those for motions $\boldsymbol{x}$. In[8] this dynamics was proposed to be named as the Oscillatory Strobodynamics (OS). The OS is the dynamics perceived by an observer with special glasses through which fast motions cannot be seen. Note that the OS may be considered also as a particular case of dynamics of systems with partially ignored motions[1,2].

Equation (5) is proposed to be named as the Equation of Oscillatory Strobodynamics[8] (EOS) or the Vibro-transformed Dynamics Equation (VDE), in contrast to equation (1) that describes "normal" dynamics.

3. The method of direct separation of motions for problems of high-frequency oscillating actions on dynamical systems

The method of direct separation of motions[1] appears to be a convenient and simple tool for obtaining equation (5). Its application for solving mechanical problems is discussed in[1,2]. Here the basics of this method for treating a broader range of problems, namely the OS problems, are described.

Inserting expression (3) into (2) we get

$$\boldsymbol{Z}[\boldsymbol{X}(t)+\boldsymbol{\psi}(t,\omega t)+\boldsymbol{\psi}_x(\omega t),\boldsymbol{a}+\boldsymbol{\psi}_{\boldsymbol{a}}(\omega t),t]=\boldsymbol{F}(\omega t)\ . \tag{6}$$

Due to the fact that we have introduced two unknown variables $\boldsymbol{X}$ and $\boldsymbol{\psi}$ instead of the initial one $\boldsymbol{x}$, we are allowed to impose an additional constraint on these variables. As this constraint we require the variables $\boldsymbol{X}$ and $\boldsymbol{\psi}$ to satisfy the averaged equation (6), so that the following relation holds true:

$$\left\langle \boldsymbol{Z}[\boldsymbol{X}(t)+\boldsymbol{\psi}(\omega t)+\boldsymbol{\psi}_x(\omega t),\boldsymbol{a}+\boldsymbol{\psi}_{\boldsymbol{a}}(\omega t),t]\right\rangle=\left\langle \boldsymbol{F}(\omega t)\right\rangle\ . \tag{7}$$

Consequently, for the initial equation (6) to be satisfied we also get:

$$\boldsymbol{Z}[\boldsymbol{X}(t)+\boldsymbol{\psi}(\omega t)+\boldsymbol{\psi}_x(\omega t),\boldsymbol{a}+\boldsymbol{\psi}_{\boldsymbol{a}}(\omega t),t]-\left\langle \boldsymbol{Z}[\boldsymbol{X}(t)+\boldsymbol{\psi}(\omega t)+\boldsymbol{\psi}_x(\omega t),\boldsymbol{a}+\boldsymbol{\psi}_{\boldsymbol{a}}(\omega t),t]\right\rangle=\boldsymbol{F}(\omega t)-\left\langle \boldsymbol{F}(\omega t)\right\rangle\ . \tag{8}$$

Combining equations (7) and (8), and taking (3) into account, we obtain initial equation (2). Note that equations (7)-(8) are not simpler than the initial equation; in particular, if operator $\boldsymbol{Z}$ represents a system of differential equations, then (7)-(8) represent a system of integro-differential equations. However, (7)-(8) are much more convenient for the approximate solving. Equations (7) and (8) are proposed to be named as the equations of slow and fast motions, respectively.

The following statements form the basis of the MDSM:

1) Slow motion $\boldsymbol{X}$ is of primary interest.

2) It is sufficient to determine variable $\boldsymbol{\psi}$ approximately, since it is present in equation (7) only under the averaging operator, and thus this approximation will not lead to any considerable errors in the resulting equation for the variable $\boldsymbol{X}$.

3) As one of the approximations, slow variables $\boldsymbol{X}$ and t are considered as constants ("frozen") when solving the fast motions equations (8).

Having determined the component $\boldsymbol{\psi}=\boldsymbol{\psi}(\boldsymbol{X},\omega t)$ from (8), and performing the averaging operation, equation (7) takes the form:

$$\boldsymbol{Z}^*(\boldsymbol{X},\boldsymbol{a}^*,t)\equiv\left\langle \boldsymbol{Z}[\boldsymbol{X}(t)+\boldsymbol{\psi}(\omega t)+\boldsymbol{\psi}_x(\omega t),\boldsymbol{a}+\boldsymbol{\psi}_{\boldsymbol{a}}(\omega t),t]\right\rangle=0\ . \tag{9}$$

As it was noted, operator $\boldsymbol{Z}^*$ can be considerably simpler than $\boldsymbol{Z}$.

The approximate approach described above implies that the velocity of component $\boldsymbol{\psi}$ variations is much larger than the velocity of $\boldsymbol{X}$ variations, i.e. component $\boldsymbol{X}$ is indeed slow, and component $\boldsymbol{\psi}$ fast. This requirement is the *main assumption of the OS*. Its mathematical description for mechanical problems is given in[1]; this description can be extended also for problems considered within the OS. The most important conditions under which the main assumption of the OS holds true are the following:

1) Frequency ω should be much larger than the characteristic frequency of slow component $\boldsymbol{X}$ variations (for applied problems three - five times larger).

2) Periodic solutions of the fast motions equations should be asymptotically stable with respect to all fast components of the vector $\boldsymbol{x}$ in the whole considered range of parameters.

The approximate method described above is based on the paper[3] by P.L. Kapitsa in which motions of a pendulum with vibrating suspension axis were considered. This method was generalized by the author, and employed by him and other scientists for solving various problems of action of vibration on mechanical systems[1,2,4,16].

4. On the validity of the MDSM

Application of the MDSM for most of the considered problems, described by differential equations, is justified by the theorems of V.M. Volosov and B.I. Morgunov[17,18]. Generalization for cases beyond these theorems is discussed in Section 5.

However, as is known (see e.g.[19]), even a theoretically justified approximate solution requires a posteriori validation. Comparison with numerical solution can be also useful. It becomes especially important for cases when strict mathematical justification is not presented or omitted. In such cases, from mathematicians' point of view, the method employed may be considered as a heuristic approach for determining solutions. If the obtained solution has been validated a posteriori, then it is considered as correct[19].

Similarly to the other approximate methods, e.g. the method of harmonic balance[15,20] and Hill's method of infinite determinants[20], the MDSM provides an explicit condition under which the obtained results are valid for every particular problem considered. Also an explicit expression which estimates the error in the obtained solution is provided.

5. Modification of the MDSM; problems without an explicit small parameter

For many practically important problems external excitation of the system cannot be considered as high-frequency, but, instead, is low-frequency, or near-resonant, or non-resonant etc. Often, response of the system to excitation from the widest possible frequency range is of interest. The modification of the MDSM discussed in the present section of the paper is for studying such cases (see also[9,13,14]).

The modified MDSM implies considering dimensionless equations, in particular the shift from the original dimensional time t to the non-dimensional one $\tau = \omega t$ is implemented. Solution is proposed to be sought in the form, similar to (3):

$$\boldsymbol{x} = \boldsymbol{X}(T_1) + \boldsymbol{\psi}(T_1, T_0), \tag{10}$$

where the new timescales $T_0 = \tau$ and $T_1 = \varepsilon T_0$ are introduced, and $\varepsilon << 1$ is a formal small parameter, and variables $\boldsymbol{X}$ and $\boldsymbol{\psi}$ have the same meaning as above. Variable $\boldsymbol{\psi}$ is 2π-periodic in time T_0, with average zero. Similarly to the multiple scales method[10], the modified MDSM implies timescales T_1 and T_0 to be considered independent.

As appears the requirement for the excitation frequency ω to be much larger than the characteristic frequency of the system's oscillations, implied in the conventional MDSM, is abandoned, $\omega \not\gg 1$. Instead of this, the restriction on the sought solutions is imposed: only solutions that are close to periodic, and describe oscillations with slowly varying amplitudes, can be determined. These solutions feature two distinct time scales, and are similar to those obtained by the classical asymptotic methods, e.g. the multiple scales method[10]. Such solutions are usually of interest for applications.

Note, however, that the introduced small parameter ε is not a feature of the considered problem or the corresponding governing equations. It is a feature of the sought solution. This constitutes the main difference of the modified MDSM from the conventional asymptotic methods.

Introducing the small parameter ε by the way described above enables to employ the modified MDSM for problems in which it is impossible to assign a small parameter in the governing equations. In particular, strongly nonlinear problems can be studied (see Subsection 6.2). The introduced small parameter ε defines proximity of the solution to pure periodic one, i.e. how slow the amplitudes are varying.

It should be noted, however, that the modified MDSM implies the conventional simplifications of the method to be abandoned. In particular, when solving fast motions equations (for variable $\boldsymbol{\psi}$), slow variables $\boldsymbol{X}$ and T_1 cannot be considered as constants.

The discussed modification of the MDSM may be considered as a development of the ideas proposed in[1] for solving equations without an explicit small parameter: A certain restriction on the sought solutions is imposed to resolve the problem. However, the problems considered in[1] implied the sought motion $\boldsymbol{x}$ to be close to motion $\boldsymbol{x}_0$ of a prescribed type, e.g. describing harmonic oscillations or uniform rotation. Consequently, the formal small parameter was introduced in front of the residual $\boldsymbol{Z}(\boldsymbol{x})-\boldsymbol{Z}(\boldsymbol{x}_0)$.

For every particular problem considered the modified MDSM provides an explicit condition under which the obtained results are valid, and estimates the error in the solution (see Subsection 6.2).

6. Examples

6.1. Systems under high-frequency oscillating actions

The conventional MDSM has been employed to study several relevant problems from various fields of science. These include analysis of vibrations of a string with pulsating tension, the problem of the so-called "Indian magic rope"[1,21], analysis of oscillating actions on Lorenz oscillator, Lotka–Volterra "predator-prey" system, Brusselator[2], and the process of nonlinear diffusion. In all these problems the system behavior changes considerably due to high-frequency periodic excitations. The details can be found in[9], see also[8].

6.2. Problems that do not imply restrictions on the spectrum of excitation frequencies

As the first example of application of the modified MDSM, we consider the classical Mathieu equation that describes oscillations arising in various mechanical, electrical and other systems, cf.[22]:

$$\frac{d^2\varphi}{dt_0^2}-\delta(1+\chi\cos t_0)\varphi=0\,. \tag{11}$$

The case of negative stiffness is studied, $\delta>0$, so that the problem of motion stabilization by means of the oscillating action is considered. Note that the equation doesn't involve a small parameter, $\delta\sim 1$, $\chi\sim 1$, and the classical asymptotic methods[10-12] cannot be used.

Employing the modified MDSM, we search a solution to (11) in the form:

$$\varphi=\alpha(T_1)+\psi(T_1,T_0)\,, \tag{12}$$

describing oscillations with slowly varying amplitudes. Here the new timescales T_1 and T_0 are defined as $T_0=t_0$, $T_1=\varepsilon T_0$; $\varepsilon<<1$ is a formal small parameter, α "slow", and ψ "fast", 2π- periodic in dimensionless time T_0 variable, with average zero. As the result, the following equation of slow motions is obtained:

$$(1-\delta\frac{\chi}{2}F_2(\delta,\chi))\frac{d^2\alpha}{dt_0^2}-\delta(1-\frac{\chi}{2}F_0(\delta,\chi))\alpha=0\,, \tag{13}$$

where functions F_i, $i=0,1,2$, are rather lengthy, and thus not given here. The details can be found in[13], see also[9]. The obtained analytical solution has been validated by the series of numerical experiments conducted by means of the Wolfram Mathematica 7.

The next problem is concerned with the analysis of the response of a nonlinear parametric amplifier. Many small-scale parametric amplifiers based on resonant micro- and nanosystems exhibit a distinctly nonlinear behavior when amplitude of their response is sufficiently large[23]. So, it becomes necessary to consider such systems dynamics in a nonlinear context, and the Duffing-type nonlinearity can serve as the simplest model. In paper[24] the near-resonant response of such system was studied for small values of the parametric excitation amplitude and the nonlinearity coefficient. Here these restrictions on the system parameters are abandoned. The governing equation is:

$$z'' + \gamma z' + \delta z + \chi z \cos 2t_0 + kz^3 = A\cos(t_0 + \phi) , \tag{14}$$

where z represents the amplifier response, γ is the coefficient of dissipation, which is assumed to be linear, A and χ are the amplitudes of the external and parametric excitations, respectively, ϕ the relative phase term, δ the squared natural frequency of the linearized system, and t_0 the dimensionless time.

Noting that $\delta \sim 1$ and $\chi \sim 1$, so that the classical asymptotic methods cannot be used, we employ the modified MDSM and compose equations of slow and fast motions. The details can be found in[14], see also[9]. Solution of the fast motions equation is sought in the form of a series

$$\psi = B_1(T_1)\cos(T_0 + \theta_1(T_1)) + B_2(T_1)\cos(2T_0 + \theta_2(T_1)) + \dots . \tag{15}$$

Influence of the second, the third and all higher harmonics on the system response for $\delta \sim 1$ and $\chi \sim 1$ turns out to be negligibly weak when either the nonlinearity coefficient k or the external excitation amplitude A is small, $k << 1$ or $A << 1$. In particular, no super- or sub-harmonic resonances can occur.

As the result, five expressions for the amplitude B_1 of the amplifier steady-state response are obtained, and stability of the solutions is predicted. Thus it is shown that the response of the nonlinear parametric amplifier features five distinct branches, three of which are stable.

Finally, to illustrate that the applicability range of the modified MDSM is not restricted to problems with non-autonomous excitation, self-excited oscillations in autonomous systems are considered for Van der Pol equation with strong nonlinearity:

$$\ddot{v} + v - \mu(1 - v^2)\dot{v} = 0 , \tag{16}$$

where parameter μ is not required to be small, $\mu \not\ll 1$. Employing the modified MDSM, we search a solution to (16) in the form of oscillations with slowly varying amplitudes:

$$v = B_{11}(T_1)\cos\lambda T_0 + B_{12}(T_1)\sin\lambda T_0 + B_{21}(T_1)\cos 3\lambda T_0 + B_{22}(T_1)\sin 3\lambda T_0 + \dots , \tag{17}$$

Here T_1 and T_0 are new timescales implied in the MDSM, $T_0 = t_0$, $T_1 = \varepsilon\lambda T_0$, and $\varepsilon << 1$ is a formal small parameter; λ is unknown frequency of self-excited oscillations to be determined.

As the result, the stationary and non-stationary behavior of the considered system has been determined. The details can be found in[9]. A series of numerical experiments was conducted to validate the obtained results in all cases showing good agreement. As an illustration, the analytically predicted dependency of the response amplitude on time t_0 is shown in Fig. 2 by the solid line for $\mu = 1$; the dashed line represents the numerical solution obtained by direct integration of the initial equation (16) using Wolfram Mathematica 7.

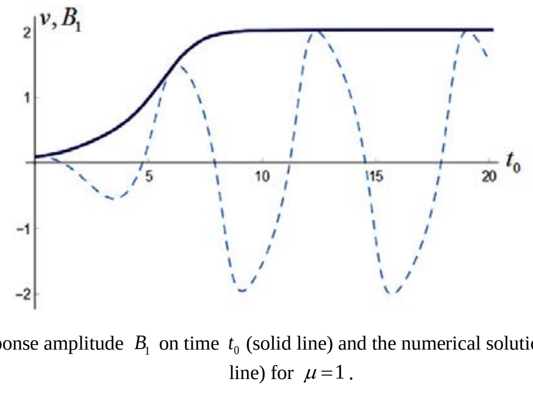

Fig. 2. The dependency of the response amplitude B_1 on time t_0 (solid line) and the numerical solution v of the initial equation (18) (dashed line) for $\mu = 1$.

7. Conclusions

The present paper considers problems of oscillating action on nonlinear dynamical systems arising in various fields of science. It is noted that such problems can be of significant applied and theoretical importance, particularly, due to the fact that generic properties of the systems can be considerably affected by the oscillating actions.

The general approach for treating problems of the considered type is proposed. This approach implies the transition from the initial governing equations of motions to equations describing only the slow component of motions which is usually of primary interest. The approach is named as the Oscillatory Strobodynamics.

The modification of the approach applicable for problems that do not imply restrictions on the spectrum of excitation frequencies is proposed. In particular, it can be employed when frequency of the oscillating action is not high, i.e. not much larger than the system characteristic (natural) frequency. The modified approach in certain cases allows also to abandon other restrictions usually introduced when employing the classical asymptotic methods, e.g. the requirement for the involved nonlinearities to be weak. So, problems without an explicit small parameter can be considered by means of the method.

The efficiency of the OS approach is illustrated in several relevant examples.

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